PROJECTED WRITTEN NOTES FROM THE M 408D LECTURE ON THURSDAY, FEBRUARY 15, 2024, ON
Power Series (centered at a) and
Power Series REPRESENTATIONS (PSRs) CLASS #10

Absolute Value and Inequalities

FACT: When $b > 0$,

$|a| < b$ means $-b < a < b$

$\le$ means $\le$

Ex: $|r| < 1$ means $-1 < r < 1$

$|x-2| < 5$ means $-5 < x-2 < 5$ (add 2

$-3 < x < 7$

$\frac{1}{5}|2x-11| < 1$

$\xrightarrow{\times 5}$ $|2x-11| < 5 \Rightarrow -5 < 2x-11 < 5$ (Add 11

$6 < 2x < 16$ [$\div$ by 2

$3 < x < 8$

Sec 11.8 Power Series

A **power series** (centered at $a=0$) is a series of the form:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

$x^0 = 1$ even when $x = 0$

Each such series defines a function $f(x)$;

$f(x) = \sum_{n=0}^{\infty} c_n x^n$ which has a domain that is equal to the set of all values of x for which The series $\sum_{n=0}^{\infty} c_n x^n$ converges.

Ex: Suppose $c_0 = c_1 = c_2 = \cdots = 1$,

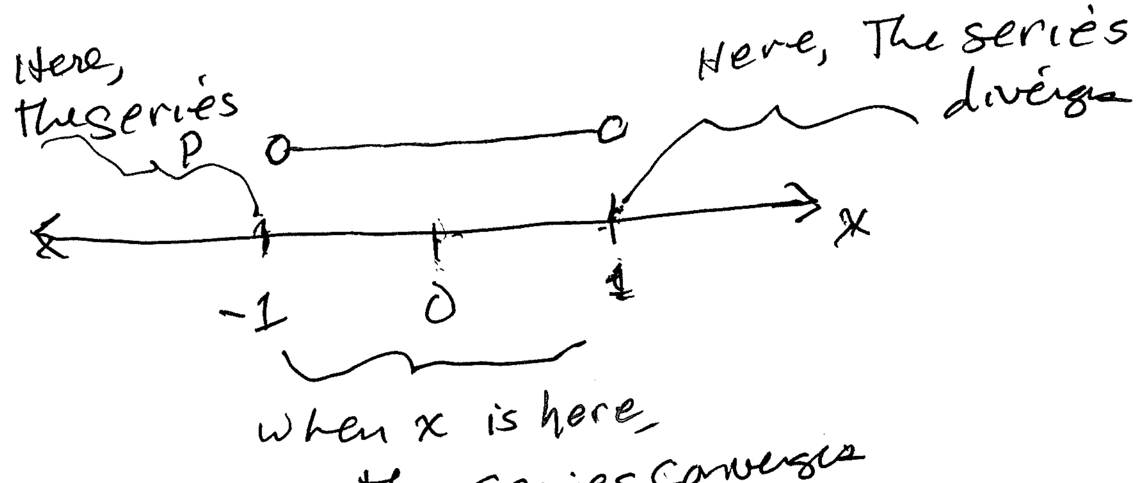
Then, $f(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$;

which is a geometric series with common ratio $r = x$.

$f(x)$ is defined and $\sum_{n=0}^{\infty} x^n$ is convergent and sums $= \frac{1}{1-x}$, only when $|r| < 1$, $-1 < r < 1$, $-1 < x < 1$.

$f(x) = \frac{1}{1-x}$ for x such that $-1 < x < 1$, $|x| < 1$.

Situation: $f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ when $|x| < 1$.



The Interval of Convergence (I.O.C.) is $(-1, 1)$, $-1 < x < 1$

More generally,
A power series "centered at a" or "at a" or "about a"

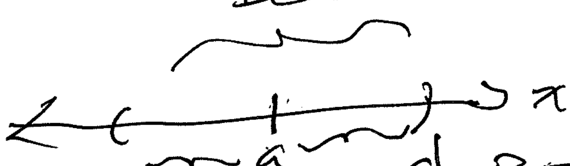
is one of the form:
$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$$

Ex: $\sum_{n=1}^{\infty} \frac{1}{n} (x-3)^n = (x-3) + \frac{1}{2}(x-3)^2 + \frac{1}{3}(x-3)^3 + \cdots$

is a power series centered at $a = 3$.

The set of values of x for which the power series converges is called the "Interval of Convergence" (IOC).

IOC $\quad \leftarrow a = \frac{c+d}{2}$



The Radius of Convergence R.

$a - c = d - a = R$ = Radius of Convergence R.

$R = \frac{d-c}{2}$

How to find the Radius R of Convergence:

Use the Ratio Test or the Root Test.

Ex: For $\sum_{n=1}^{\infty} c_n(x-a)^n = \sum_{n=1}^{\infty} \frac{1}{n}(x-3)^n$, look at $\frac{|a_{n+1}|}{|a_n|}$.

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\left|\frac{1}{n+1}(x-3)^{n+1}\right|}{\left|\frac{1}{n}(x-3)^n\right|} = \left(\frac{1}{n+1}|x-3|^{n+1}\right)\left(\frac{n}{1}\cdot\frac{1}{|x-3|^n}\right)$$

$$= \frac{n}{n+1}|x-3|$$

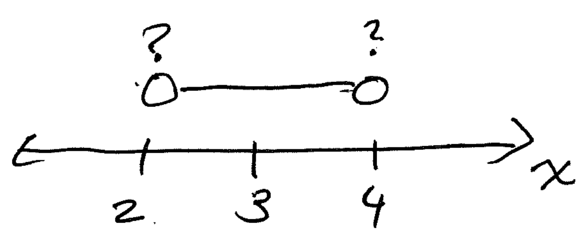
x is fixed.

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \underbrace{\left(\frac{n}{n+1}\right)}_{\to 1}|x-3| = |x-3| = L$$

We need $L < 1$; so, $L = |x-3| < 1$

$-1 < x-3 < 1 \xrightarrow[\text{add } 3]{} 2 < x < 4$ This is only part of the I.O.C.

center $= \frac{4+2}{2} = 3$. $R = \frac{4-2}{2} = 1$ $a=3$, $R=1$



We must check for series convergence at the IOC endpoints. Here, at $x=2$ and at $x=4$.

The Power Series is $\sum_{n=1}^{\infty} \frac{1}{n}(x-3)^n$.

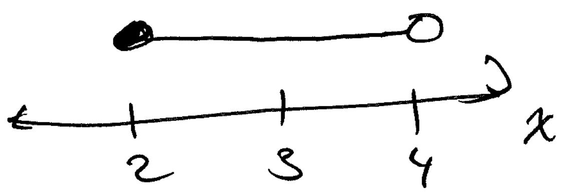
At $x=2$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{n}\underbrace{(2-3)^n}_{(-1)^n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$

which is **convergent** by the Alternating Series Test.

At $x=4$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{n}\underbrace{(4-3)^n}_{1^n} = \sum_{n=1}^{\infty} \frac{1}{n}$,

which is **Divergent** since it is the Harmonic Series.

For $\sum_{n=1}^{\infty} \frac{1}{n}(x-3)^n$, The I.O.C is $2 \le x < 4$ or $[2,4)$.



Interval of Convergence

For the power series that we have looked at so far, the set of values of x for which the series is convergent has always turned out to be an interval [a finite interval for the geometric series and the series in Example 1, the infinite interval $(-\infty, \infty)$ in Example 3, and a collapsed interval $[0, 0] = \{0\}$ in Example 2]. The following theorem, proved in Appendix F, says that this is true in general.

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4 Theorem For a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities:

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.

The number R in case (iii) is called the **radius of convergence** of the power series. By convention, the radius of convergence is $R = 0$ in case (i) and $R = \infty$ in case (ii). The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges. In case (i) the interval consists of just a single point a. In case (ii) the interval is $(-\infty, \infty)$. In case (iii) note that the inequality $|x - a| < R$ can be rewritten as $a - R < x < a + R$. When x is an *endpoint* of the interval, that is,

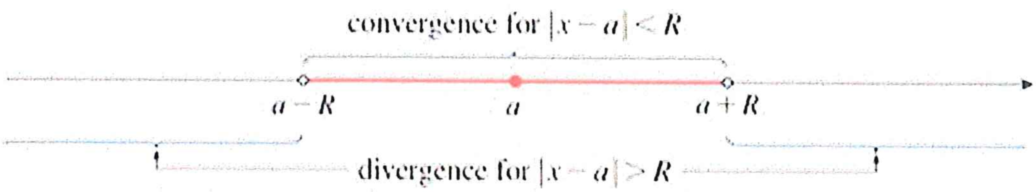

FIGURE 1

We summarize here the radius and interval of convergence for each of the examples already considered in this section.

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	Series	Radius of convergence	Interval of convergence
Geometric series	$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
Example 1	$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$
Example 2	$\sum_{n=0}^{\infty} n!\, x^n$	$R = 0$	$\{0\}$
Example 3	$\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$	$R = \infty$	$(-\infty, \infty)$

NOTE In general, the Ratio Test (or sometimes the Root Test) should be used to determine the radius of convergence R. The Ratio and Root Tests always fail when x is an endpoint of the interval of convergence, so the endpoints must be checked with some other test.

Algebra Discussion

Recall that $\sqrt{x^2} = x$ only when $x \geq 0$.

$\sqrt{x^2} = |x|$ always

Simplifying Power Series

(1) Drop all initial terms equal to zero.

$$\sum_{n=0}^{\infty} \underbrace{n(n-1)}(x-3)^n = \sum_{n=2}^{\infty} n(n-1)(x-3)^n.$$

$=0$ when $n=0$ and $n=1$, so start at $n=2$

(2) Rewrite the power series as $\sum_{n=?}^{\infty} c_n x^n$, if possible, i.e., with x^n rather than x^{n-1} or x^{n+2}, ---

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Summation Rewriting Procedure

Given $\sum_{n=1}^{\infty} \frac{1}{7^{n+1}} n x^{(n-1)}$, Replace x^{n-1} with x^n, do this!

Goal: Replace $(n-1)$ with n
$\quad\quad\quad\quad\quad\quad +1 \quad\quad\quad\quad +1$

Task: Replace n: with $(n+1)$, Everywhere

Task: Replace n with $n+1$:

$$\sum_{\substack{n+1=1 \\ -1 \quad -1 \\ n=0}}^{\infty} \frac{1}{7^{(n+1)+1}} (n+1)\, x^{(n+1)-1} = \sum_{n=0}^{\infty} \frac{1}{7^{n+2}} (n+1)\, x^{n}$$

Power Series Representations (PSRs)

Defn: Given a function $f(x)$ and a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, (centered at a)

We say "The power series is a Power Series Representation of $f(x)$" if, for all x in the IOC of the power series,

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad \leftarrow \text{Centered at } a.$$

when $a=0$, this is $f(x) = \sum_{n=0}^{\infty} c_n x^n$ ← centered at $a=0$.

Example: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ when $|x| < 1$.

$c_0 = c_1 = c_2 = \dots = 1$, Geometric Series with $r=x$,

First term is 1, So, $f(x) = \frac{1}{1-x}$.

$f(x) = \sum_{n=0}^{\infty} x^n$, $|x| < 1$, is a PSR of $f(x) = \frac{1}{1-x}$

with Radius of Convergence $R=1$.

Note: With PSRs, we don't need to find the complete IOC. We only need to state the power series and its Radius R.

Problem: (Showing Method ② from the Handout).

Determine a PSR for $f(x) = \dfrac{1}{3+x^2}$ and find R, the Radius of Convergence.

Sol'n: $f(x) = \dfrac{1}{3+x^2} = \dfrac{1}{3\left(1+\frac{x^2}{3}\right)} = \dfrac{1}{3\left(1-\left(-\frac{x^2}{3}\right)\right)}$

$$= \frac{1}{3}\left(\frac{1}{1-\left(-\frac{x^2}{3}\right)}\right)$$

The Sum of a Geometric series with $r = -\frac{x^2}{3}$

$$= \frac{1}{3}\left(\sum_{n=0}^{\infty}\left(-\frac{x^2}{3}\right)^n\right)$$

$$= \frac{1}{3}\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{3^n}\right)$$

$$f(x) = \frac{1}{3+x^2} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{3^{n+1}},$$ and $R=\sqrt{3}$ (see below)

FINDING R: Here $|r| < 1 \Rightarrow \left|-\frac{x^2}{3}\right| < 1 = \frac{|x|^2}{3} < 1$

So, (multiplying by 3), $|x|^2 < 3 \Rightarrow |x| < \sqrt{3}$, $\underline{R = \sqrt{3}}$

$|x-0| < \sqrt{3}$